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The theory of electrostatic potential in superconductors

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Abstract. The problem of electrostatic potential distribution in superconductors is studied based on the microscopic theory of superconductors. Local chemical potential of the superconductor is introduced, and an approximation is made to BG theory, which is similar to the Thomas-Fermi (TF) method used in quantum mechanics. The electrostatic potential and charge distribution around an isolated vortex in type-II superconductors is discussed within the approximation. A correction to GL theory considering the electrostatic potential distribution is suggested.

PACS. 74.20.Fg BCS theory and its development – 74.20.De Phenomenological theories (two-fluid, Ginzburg-Landau, etc.) – 74.25.Jb Electronic structure

1 Introduction

It was predicted a long time ago by London [1] that an electric field would occur in superconductors where the distribution of superconducting current is inhomogeneous, which is similar to the Bernoulli effect in fluid mechanics. Here the electrostatic potential is to compensate the difference of chemical potential of charge carriers caused by the different kinetic energy of superconducting charge carriers in different parts of the superconductor having different current density, so that the chemical potential of the whole system would be identical [2,3].

After the discovery of high temperature superconductor, the problem of electrostatic potential distribution in superconductors has been discussed in many papers [4,5]. Khomskii and Freimuth [6] discussed the problem of an isolated vortex carrying charge, emphasizing that the chemical potential near the core is higher than that far away from the vortex. Koláček, Lipavský and Brandt [7] discussed the same problem. They emphasized that in a superconductor where the distribution of order parameter is inhomogeneous, the density of both normal and superconducting electrons are inhomogeneous, and the diffusion of the superconducting and normal electrons together with the Lorentz force should be compensated by a distribution of electric potential. In a recent paper [8], Lipavský, Koláček, Morawetz and Brandt developed this theory. They extended Ginzburg-Landau (GL) theory based on a generalized two-fluid model, in order to solve the problem of electrostatic potential in superconductors where the distribution of order parameter is inhomogeneous. In this paper [8], they also gave a detailed review to the past works on the electrostatic potential distribution in superconductors. The experimental method to detect the electrostatic potential in superconductors has been discussed in some papers [9–11]. Yampolskii et al. [11] also discussed the electric potential distribution in mesoscopic superconductors.

In the present paper, we will derive a new method based on the microscopic theory of superconductivity to deal with the electrostatic potential in superconductors where the distribution of order parameter is inhomogeneous. This method is to make an approximation to the Bogoliubov-de Gennes(BG) self-consistent theory of superconductors [12], which is similar to the Thomas-Fermi (TF) method used in quantum mechanics, and we have used the results of GL theory in the discussion. As is well known, BG self-consistent theory is equivalent to Gorkov's microscopic theory of Green's functions [13]. Deriving GL theory from Green's functions is also equivalent to deriving from BG theory [14,15]. In this paper, we will firstly give a brief description to BG theory using the treatment of de Gennes [12]. Then we discuss the superconductors where the distribution of order parameter is inhomogeneous and introduce a local chemical potential using a TF-like approximation and the results of GL theory. We derive the expression of electrostatic potential distribution based on the principle that the chemical potential in different parts of the superconductor should be identical. Then we reconsider the problem of an isolated vortex in type-II superconductors and discuss the distribution of electrostatic potential and electric charge around the vortex core. Finally we will discuss the correction to GL

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theory when considering the electrostatic potential distribution.

2 BG self-consistent theory

BG theory is actually the generalization of Hartree-Fock self-consistent theory for superconductors. In BG theory, the problem of superconductor with magnetic field becomes a problem of deriving self-consistent solution to the eigenfunctions. For a s-wave superconductor, the Hamiltonian in BG theory may be written as

$$\mathscr{H}_{BG} = \begin{bmatrix} \mathscr{H}_n + \delta U - \mu_s & \Delta \\ \Delta^* & -[\mathscr{H}_n^* + \delta U - \mu_s] \end{bmatrix}. \quad (1)$$

Here $\mathscr{H}_n$ is the self-consistent Hamiltonian of the charge carrier in normal state

$$\mathscr{H}_n = \frac{1}{2m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2 + U_n \quad (2)$$

where $\mathbf{A}$ is the vector potential of the magnetic field. U_n is self-consistent field of the charge carrier, including the field of crystal lattice and the interaction between charge carriers. δU is the change of the self-consistent potential from normal state to superconducting state, and μ_s is the chemical potential of the system in superconducting state. BG equation is written as

$$\mathscr{H}_{BG}\begin{bmatrix} u_k \\ v_k \end{bmatrix} = E_k \begin{bmatrix} u_k \\ v_k \end{bmatrix}. \quad (3)$$

The density of charge carrier in superconducting state is written as

$$\rho_s = \sum_k \left[|u_k(\mathbf{r})|^2 f_k + |v_k(\mathbf{r})|^2(1 - f_k)\right] \quad (4)$$

where $f_k = 1/(e^{\beta E_k} + 1)$, $\beta = 1/k_B T$ and $E_k \geq 0$. If the spatial variation of the order parameter and the magnetic field is slow, we may assume the gap operator $\Delta(\mathbf{r}, \mathbf{r}') \simeq \Delta(\mathbf{r})$, and it is expressed as

$$\Delta(\mathbf{r}) = V \sum_k v_k^*(\mathbf{r}) u_k(\mathbf{r})(1 - 2f_k). \quad (5)$$

The change of self-consistent potential δU is determined by

$$\delta U(\mathbf{r}) = \int V(\mathbf{r}, \mathbf{r}')[\rho_s(\mathbf{r}') - \rho_n(\mathbf{r}')]d\mathbf{r}'. \quad (6)$$

Chemical potential μ_s is determined by

$$\int (\rho_s - \rho_n) d^3 r = 0 \quad (7)$$

where ρ_n is the density of charge carrier in normal state. In principle, given μ_s, $\delta U(\mathbf{r})$ and $\Delta(\mathbf{r})$, one has u_k, v_k and E_k from (3). Then one can solve self-consistent μ_s, $U(\mathbf{r})$ and $\Delta(\mathbf{r})$ from equations (4–6), and one may repeat the procedures to achieve higher precision, until $\Delta(\mathbf{r})$ and $\delta U(\mathbf{r})$ are fully consistent. But since the $\Delta(\mathbf{r})$ is nonzero only in the neighborhood of Fermi surface, the change of density of charge carrier is small. Thus we may introduce some approximations to the BG theory.

3 TF-like approximation

We know that the density of state in the neighborhood of the Fermi energy is not exactly symmetric to the Fermi surface. Thus the chemical potential of the superconducting state μ_s is in fact different from the chemical potential of normal state μ_n under same temperature and electron density [16]. Now we derive this difference explicitly from The BG theory. For a homogeneous system in the absence of magnetic field, BG equation (3) could be solved as [12]

$$E_k = \sqrt{(\varepsilon - \mu_s)^2 + |\Delta|^2} \quad (8a)$$

$$|u_k|^2 = \frac{1}{2}\left(1 + \frac{\varepsilon - \mu_s}{E_k}\right) \quad (8b)$$

$$|v_k|^2 = \frac{1}{2}\left(1 - \frac{\varepsilon - \mu_s}{E_k}\right) \quad (8c)$$

where $\varepsilon = \mathscr{H}_n + \delta U$ is the energy of the charge carrier. The density of charge carrier (4) may be re-written as

$$\rho_s = \int_0^\infty [|u_k|^2 f_k + |v_k|^2(1 - f_k)]\mathscr{D}(\varepsilon)d\varepsilon \quad (9)$$

where $\mathscr{D}(\varepsilon)$ is the density of energy state of electrons in normal state. Apparently, for homogeneous systems in the absence of magnetic field, the density of charge carrier is always $\rho_s = \rho_n$ and does not change spatially. From (9) we may deduce the expression of chemical potential of this homogeneous system in superconducting state as (see Appendix A)

$$\mu_s(\Delta, T) \simeq \mu_n - \frac{1}{2}|\Delta|^2 \frac{d \ln \mathscr{D}_F}{d\varepsilon} \mathscr{R}(\Delta, T) \quad (10)$$

where $\mathscr{D}_F = \mathscr{D}(\mu_n)$ is the density of energy state at Fermi surface for normal state. μ_n is the Fermi energy of normal state at $T = 0$. $\mathscr{R}(\Delta, T)$ is a parameter. When $T = 0$, $\mathscr{R}(\Delta, T)$ could be written as

$$\mathscr{R}_0(\Delta) \simeq \ln \frac{2\hbar\omega_D}{|\Delta|} - 1 \quad (11)$$

where ω_D is the Debye frequency. When $T > 0$, the parameter $\mathscr{R}$ will change, but (10) is still approximatively tenable.

For superconductors where the order parameter is inhomogeneous, the order parameter and the density of charge carrier becomes spatial functions: $\Delta = \Delta(\mathbf{r})$ and $\rho_s = \rho_s(\mathbf{r})$. If the outer field is not very strong and is restricted to part of the space, and if the spatial change of the order parameter $\Delta(\mathbf{r})$ is slow, we may regard the neighborhood of $\mathbf{r}$ as a small subsystem whose temperature is T, whose superconducting electron density is $\rho_s(\mathbf{r})$ and whose order parameter is $|\Delta(\mathbf{r})|$. Since the subsystem is very small, both $\rho_s(\mathbf{r})$ and $|\Delta(\mathbf{r})|$ can be regarded as constant in the subsystem. That is, the small subsystem in the inhomogeneous superconductor can be regarded as *homogeneous*. Thus we may use the result of (10) by defining a local chemical potential μ_L for this subsystem. In principle, the chemical potential should be strictly written as $\mu_L = \mu_L(\Delta, T, \rho_s(\mathbf{r}))$ and $\mu_n = \mu_n(\rho_s(\mathbf{r}))$. But we have pointed out that ρ_s is very close to ρ_n. So we may take $\rho_s \simeq \rho_n$ and use the result of (10) to write the local chemical potential as

$$\mu_L(\Delta, T) \simeq \mu_n - \frac{1}{2}|\Delta(\mathbf{r})|^2 \frac{d \ln \mathscr{D}_F}{d\varepsilon} \mathscr{R}(\Delta(\mathbf{r}), T). \quad (12)$$

For a place deep inside the superconductor where the magnetic field is almost zero, the local chemical potential μ_L comes back to μ_s, now expressed as μ_∞

$$\mu_L(\Delta(\mathbf{r}))|_\infty \to \mu_\infty \quad (13)$$

where

$$\mu_\infty \simeq \mu_n - \frac{1}{2}|\Delta_\infty|^2 \frac{d \ln \mathscr{D}_F}{d\varepsilon} \mathscr{R}_\infty. \quad (14)$$

Here $\mathscr{R}_\infty = \mathscr{R}(\Delta_\infty, T)$. Δ_∞ is the order parameter deep inside the superconductor. It equals to the order parameter of the superconductor in the absence of magnetic field. The difference between the local chemical potential at $\mathbf{r}$ and the chemical potential far away is denoted with $\delta\mu_L = \mu_L - \mu_\infty$ and is expressed as

$$\delta\mu_L \simeq -\frac{1}{2} \frac{d \ln \mathscr{D}_F}{d\varepsilon} [|\Delta|^2 \mathscr{R}(\Delta) - |\Delta_\infty|^2 \mathscr{R}_\infty]. \quad (15)$$

When $T = 0$, we can use the expression (11) to calculate $\delta\mu_L$. When $T > 0$, the value of $\mathscr{R}$ will change, but (15) is still valid. It should be noticed that the effect of charge screening has been taken fully account into (15). A brief discussion to this point is made in Appendix B.

Up to now we have only discussed the subsystems whose order parameter is $|\Delta(\mathbf{r})|$. But the order parameter of an inhomogeneous superconductor is normally complex, and it also has nonzero spatial gradient which causes superconducting current. Now we discuss these two parts using GL theory. The GL free energy density of the superconducting state relative to normal state is

$$\mathscr{F} = -A|\Delta|^2 + \frac{1}{2}B|\Delta|^4 + C\left|\left(\nabla - i\frac{2e}{\hbar c}\mathbf{A}\right)\Delta\right|^2 + \frac{h^2}{8\pi} \quad (16)$$

where h denotes the magnetic field. For reduced BCS model, when $T \sim T_c$, A, B and C could be written as [12]

$$A \simeq \mathscr{D}_F \left(1 - \frac{T}{T_c}\right) \quad (17a)$$

$$B \simeq \mathscr{D}_F \frac{1}{\Delta_c^2} \quad (17b)$$

$$C \simeq \frac{1}{6}\mathscr{D}_F \left(\frac{\hbar v_F}{\Delta_c}\right)^2 \quad (17c)$$

where v_F is the electron velocity at Fermi surface. When $T \sim T_c$ and in the absence of a magnetic field, Δ_c is determined by

$$\Delta_\infty(T) \simeq \Delta_c \sqrt{(1 - T/T_c)}. \quad (18)$$

According to reduced BCS theory, we have $\Delta_c \simeq 3.2 k_B T$. According to the thermodynamic formulae [19], we know that the change of chemical potential from normal to superconducting state caused by the spatial gradient of order parameter and the superfluid of electrons is determined by

$$\mu_B(\mathbf{r}) = \frac{\partial \mathscr{F}}{\partial \rho_n}. \quad (19)$$

Notice that $A, B \propto \mathscr{D}_F$, i.e. $A, B \propto k_F \propto \rho_n^{1/3}$, where k_F is the Fermi wave vector. And we can see that $C \propto \mathscr{D}_F v_F^2$, which means $C \propto k_F^3 \propto \rho_n$. Hence when $\rho_n \to \infty$, the items contains A and B respectively would vanish during the derivation according to ρ_n, and only the part containing C would play the main role in equation (19). Thus we may write μ_B as

$$\mu_B(\mathbf{r}) \simeq \frac{\partial C}{\partial \rho_n}\left|\left(\nabla - i\frac{2e}{\hbar c}\mathbf{A}\right)\Delta\right|^2. \quad (20)$$

As we can see, μ_B contains the effect of both spatial gradient of the order parameter and the supercurrent. Accordingly, the difference between the local chemical potential for a subsystem in the neighborhood of $\mathbf{r}$ and the chemical potential far away is expressed as

$$\delta\mu(\mathbf{r}) = \delta\mu_L(\mathbf{r}) + \mu_B(\mathbf{r}) \quad (21)$$

and this difference should be compensated with an electrostatic potential relevant to $\mathbf{r}$, in order to keep the chemical potential identical in different parts of the whole system. This electrostatic potential is determined by

$$\varphi = -\frac{1}{q}\delta\mu(\mathbf{r}). \quad (22)$$

Here q is the charge of the current carrier.

Now we have deduced the electrostatic potential φ, which can be regarded as the first order approximation of the δU. The method we have used is similar to the Thomas-Fermi method solving many-body problems in quantum mechanics. Or we may say that we have made a TF approximation to BG self-consistent theory. As is shown in Appendix B, the charge screening effect has been taken into account by a step by step approximation with respect to the small parameter l_{cs}^2/ξ^2, where l_{cs} is the charge screening length and ξ is the coherence length.

In principle, we should substitute the electric potential (22) back into BG equations (3) and do further calculations in order to get self-consistency. But let us first estimate the order of magnitude of $\delta\mu(\mathbf{r})$ and see whether further calculation is necessary.

4 The estimate to the change of chemical potential

To estimate the order of magnitude of $\delta\mu(\mathbf{r})$, we have to make some further approximations. First we write $\delta\mu$ in an expanded form

$$\delta\mu = -\frac{1}{2}\frac{d\ln\mathscr{D}_F}{d\varepsilon}[|\Delta|^2\mathscr{R}(\Delta,T) - |\Delta_\infty|^2\mathscr{R}(\Delta_\infty,T)] + \frac{\partial C}{\partial\rho_n}\left|\left(\nabla - i\frac{2e}{\hbar c}\mathbf{A}\right)\Delta\right|^2. \quad (23)$$

Now we omit parameter T and write $\mathscr{R}(\Delta,T)$ as $\mathscr{R}(\Delta)$, write $\mathscr{R}(\Delta_\infty,T)$ as $\mathscr{R}_\infty$. And we denote $\delta\mu_L$ with $\delta\mu_1$, which means the first part of (23)

$$\delta\mu_1 \equiv -\frac{1}{2}\frac{d\ln\mathscr{D}_F}{d\varepsilon}[|\Delta|^2\mathscr{R}(\Delta) - |\Delta_\infty|^2\mathscr{R}_\infty].$$

When $T = 0$, we have

$$\begin{aligned}|\Delta|^2\mathscr{R}(\Delta) &= |\Delta|^2\left(\ln\frac{2\hbar\omega_D}{|\Delta|} - 1\right)\\ &= |\Delta|^2\left(\ln\frac{2\hbar\omega_D}{|\Delta_\infty|} + \ln\frac{|\Delta|}{|\Delta_\infty|} - 1\right)\\ &= |\Delta|^2\mathscr{R}_\infty - |\Delta|^2\ln\frac{|\Delta|}{|\Delta_\infty|}.\end{aligned}$$

So $\delta\mu_1$ is written as

$$\delta\mu_1 = -\frac{1}{2}\frac{d\ln\mathscr{D}_F}{d\varepsilon}\left[(|\Delta|^2 - |\Delta_\infty|^2)\mathscr{R}_\infty + |\Delta|^2\ln\frac{|\Delta|}{|\Delta_\infty|}\right].$$

When $T = 0$, $\mathscr{R}_\infty$ is determined by (11). When $T > 0$, the value of $\mathscr{R}_\infty$ will change, but we can still use the expression of (11) for rough estimation. Apparently, we can see that $0 < |\Delta|^2\ln\frac{|\Delta|}{|\Delta_\infty|} < |\Delta_\infty|^2$ And when $|\Delta| \to 0$ or $|\Delta| \to |\Delta_\infty|$, we have $|\Delta|^2\ln\frac{|\Delta|}{|\Delta_\infty|} \to 0$. On the other hand, we also have $0 < |\Delta|^2 < |\Delta_\infty|^2$. Thus we may estimate the order of magnitude of $\delta\mu_1$ like

$$\delta\mu_1 \sim \frac{1}{2}\frac{d\ln\mathscr{D}_F}{d\varepsilon}|\Delta_\infty|^2\left(\ln\frac{2\hbar\omega_D}{|\Delta_\infty|} - 1\right). \quad (24)$$

When $T > 0$, the value of $\delta\mu_1$ will change, but it will remain in similar order. So we may use (24) to estimate the order of $\delta\mu_1$ as a rough approximation.

For reduced BCS model, we have [17]

$$\ln\frac{2\hbar\omega_D}{|\Delta_\infty|} \sim \frac{1}{\mathscr{D}_F\mathscr{V}} \quad (25)$$

where $\mathscr{V}$ is the electron-photon interaction parameter. For real material, $1/\mathscr{D}_F\mathscr{V} \sim (2-5)$ [17]. We can also estimate that

$$\frac{d\ln\mathscr{D}_F}{d\varepsilon} \sim \frac{\mathscr{C}}{E_F} \quad (26)$$

where E_F is Fermi energy and $\mathscr{C}$ is a constant having order of 1. Hence $\delta\mu_1$ has the order of $|\Delta_\infty|^2/E_F$.

The second part of (23) is μ_B. When using reduced BCS theory and free electron gas model, we have

$$\frac{\partial C}{\partial\rho_n} \simeq \frac{\hbar^2}{4m}\frac{1}{\Delta_c^2}. \quad (27)$$

Suppose we can write Δ as $\Delta = \Delta_r(x)e^{i\theta(x)}$, in which Δ_r is real, and substitute it into μ_B

$$\begin{aligned}\delta\mu_B &= \frac{\hbar^2}{4m}\frac{1}{\Delta_c^2}\left|\left(\nabla - i\frac{2e}{\hbar c}\mathbf{A}\right)\Delta\right|^2\\ &= \frac{\hbar^2}{4m}\frac{1}{\Delta_c^2}\left[(\nabla\Delta_r)^2 + \Delta_r^2\left|\nabla\theta - \frac{2e}{\hbar c}\mathbf{A}\right|^2\right]\\ &= \delta\mu_2 + \delta\mu_3. \end{aligned} \quad (28)$$

Here $\delta\mu_2$ and $\delta\mu_3$ are defined as

$$\delta\mu_2 \equiv \frac{\hbar^2}{4m}\frac{1}{\Delta_c^2}(\nabla\Delta_r)^2 \quad (29)$$

$$\delta\mu_3 \equiv \frac{\hbar^2}{4m}\frac{1}{\Delta_c^2}\Delta_r^2\left|\nabla\theta - \frac{2e}{\hbar c}\mathbf{A}\right|^2. \quad (30)$$

The spatial gradient of $\Delta(\mathbf{r})$ has the order of ξ^{-1}. Hence $(\nabla\Delta_r)^2 \sim \Delta_c^2\xi^{-2}$, where ξ is the coherent length. $\Delta_r^2\left|\nabla\theta - \frac{2e}{\hbar c}\mathbf{A}\right|^2$ is the effect of superconducting current, which was formally considered as Bernoulli effect that originally suggested by London et al. [1–3]. We will prove that this part has the same order of magnitude with $(\nabla\Delta_r)^2$. Since $mv_F^2/2 \sim E_F$ and $\xi^{-2} \sim \pi^2\Delta_\infty/(\hbar v_F)^2$, $\delta\mu_2$ also has the order of $|\Delta_\infty|^2/E_F$.

Up to now we have shown that the change of chemical potential from $\mathbf{r}$ to a place deep inside the superconductor consists of three parts: $\delta\mu_1$, $\delta\mu_2$ and $\delta\mu_3$. The first part $\delta\mu_1$ is determined by the change of the order parameter, $\delta\mu_1 \propto |(\Delta_\infty|^2 - |\Delta|^2)$. The second part $\delta\mu_2$ is determined by the spatial gradient of the mold of order parameter, $\delta\mu_2 \propto (\nabla\Delta_r)^2$. The third part $\delta\mu_3$ contains the spatial gradient of the phase of the order parameter, representing the effect of the change of supercurrent density, which was formally regarded as Bernoulli effect. All three parts have similar order of magnitude, and none of them should be neglected.

The additional density of charge carrier ρ_{ad} is determined by $\rho_s = \rho_n + \rho_{ad}$. It can be written as

$$\rho_{ad} = -\frac{1}{4\pi q}\nabla^2\varphi. \quad (31)$$

Since we have shown that $\delta\mu = \delta\mu_1 + \delta\mu_2 + \delta\mu_3$ has the order if $|\Delta_\infty|^2/E_F$, and the spatial gradient of $\Delta(\mathbf{r})$ has the order of ξ^{-1}, we may estimate the order of ρ_{ad} as

$$\rho_{ad} \sim \frac{1}{4\pi\xi^2 q}\frac{|\Delta_\infty|^2}{E_F}. \quad (32)$$

We know that $|\Delta| \ll E_F$ and $E_F \sim 1$ eV. Since normally we have $\rho_n \sim 10^{22}$ in superconductors, we can see that $\rho_{ad} \ll \rho_n$. So $\rho_n \simeq \rho_s$ can be taken as a good approximation.

5 Electrostatic potential of an isolated vortex

Now we reconsider the problem of an isolated vortex in a type-II superconductor using the result above. Taking cylindrical coordinate $(r, \theta, \mathbf{z})$ with zero point at the core of the vortex, we may write the order parameter, vector potential and magnetic field as

$$\Delta(r) = \Delta_\infty f(r) e^{i\theta} \tag{33a}$$
$$\mathbf{A} = a(r)\mathbf{e}_\theta \tag{33b}$$
$$\mathbf{H} = h(r)\mathbf{e}_z. \tag{33c}$$

Here Δ_∞ is taken real. This problem has been well studied in many books. We also give a brief discussion about the problem of an isolated vortex in a type-II superconductor in Appendix C. From Appendix C we have

$$f(r) \simeq \tanh(\nu r/\xi) \tag{34a}$$
$$a(r) \simeq \frac{\hbar c}{2e\xi}\frac{1}{r}\left[1 - \frac{yK_1(y/\lambda)}{y_0 K_1(y_0/\lambda)}\right] \tag{34b}$$
$$H \simeq \frac{\Phi_0}{2\pi\lambda^2}K_0(y/\lambda) \tag{34c}$$

where $y = \sqrt{y_0^2 + r^2}$. ν and y_0 are constant. When $\kappa \gg 1$, we have $\nu \simeq 0.61$, $y_0 \simeq 1.63\xi$. Substituting (33) and (34) into $\delta\mu$ expressed by (23), we may write $\delta\mu_s$ as the sum of three parts

$$\delta\mu_s = \delta\mu_{v1} + \delta\mu_{v2} + \delta\mu_{v3} \tag{35}$$

where

$$\delta\mu_{v1} = -\frac{1}{2}\frac{d\ln\mathscr{D}_F}{d\varepsilon}[|\Delta|^2\mathscr{R}(\Delta) - |\Delta_\infty|^2\mathscr{R}_\infty] \tag{36a}$$
$$\delta\mu_{v2} = \frac{\hbar^2}{4m}\frac{|\Delta|^2}{\Delta_c^2}\left(\frac{df}{dr}\right)^2 \tag{36b}$$
$$\delta\mu_{v3} = \frac{\hbar^2}{4m}\frac{|\Delta|^2}{\Delta_c^2}f^2(r)\left[\frac{1}{r} - \frac{2e}{\hbar c}a(r)\right]^2. \tag{36c}$$

Using the results of Section 4, we may roughly calculate the order and the shape of the electrostatic potential and the charge distribution. $\delta\mu_{v1}$ can estimated as

$$\delta\mu_{v1} \sim \frac{1}{2}\frac{\mathscr{C}}{E_F}|\Delta_\infty|^2\left(\frac{1}{\mathscr{D}_F\mathscr{V}} - 1\right)\cosh^{-2}\left[\frac{\nu r}{\xi}\right] \tag{37}$$

$\delta\mu_{v1}$ is nearly a constant when $r \le \xi$, and approaches $e^{-2\nu r/\xi}$ when $r \gg \xi$.

$\delta\mu_{v2}$ contains the spatial gradient of the module of order parameter. It has the expression of

$$\delta\mu_{v2} \simeq \frac{\hbar^2}{4m}\left(1 - \frac{T}{T_c}\right)\frac{\nu^2}{\xi^2}\cosh^{-4}\left(\frac{\nu r}{\xi}\right) \tag{38}$$

$\delta\mu_{v2}$ equals to $\frac{\hbar^2\nu^2}{4m\xi^2}\left(1 - \frac{T}{T_c}\right)$ when $r = 0$, and it has the same order with $\delta\mu_{v1}$. $\delta\mu_{v2}$ is nearly a constant when $r \le \xi$, and approaches $e^{-4\nu r/\xi}$ when $r \gg \xi$.

$\delta\mu_{v3}$ contains the contribute of the supercurrent, and this part was formally considered as Bernoulli effect. It is written as

$$\delta\mu_{v3} \simeq \frac{\hbar^2}{4m}\left(1 - \frac{T}{T_c}\right)\frac{f^2(r)}{r^2\xi^2}\frac{yK_1(y/\lambda)}{y_0K_1(y_0/\lambda)} \tag{39}$$

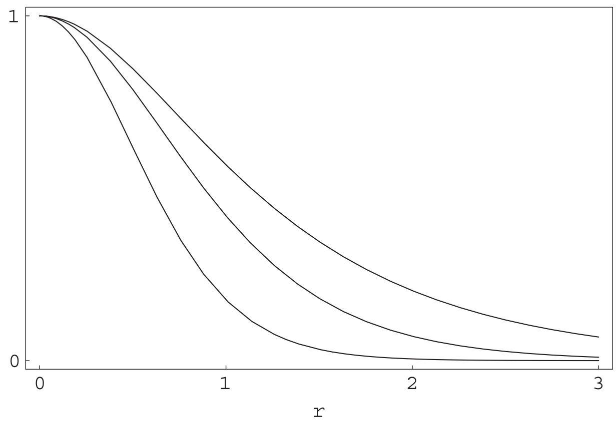

Fig. 1. The variation of $\delta\mu_{v1}$ (the middle curve), $\delta\mu_{v2}$ (the lower curve) and $\delta\mu_{v3}$ (the upper curve) with r. Please notice that the value of the three parts of $\delta\mu$ has been normalized. The unit of the x coordinate is $\xi = 1$.

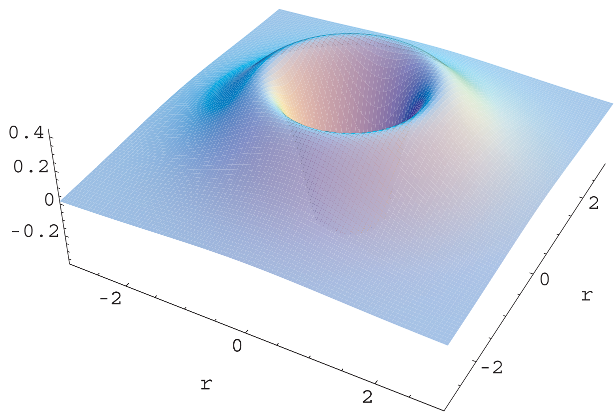

Fig. 2. The charge density around an isolated vortex line where the charge carrier is hole. The unit of the coordinate is $\xi = 1$. We can see that the column containing net charge has a diameter of $r \sim \xi$. A "shell" of positive charge can be seen surrounding the negative column and will fade out when $r \sim \lambda$.

$\delta\mu_{v3}$ equals to $\frac{\hbar^2}{4m\xi^2}\left(1 - \frac{T}{T_c}\right)$ when $r = 0$ and is nearly a constant when $r \le \xi$. When $\xi < r < \lambda$, $\delta\mu_{v3} \propto r^{-2}$. When $r > \lambda$, $\delta\mu_{v3}$ approaches $e^{-2r/\lambda}$.

The variation of the three parts of $\delta\mu$ discussed above according to r is shown in Figure 1, and the charge density caused by the additional electrostatic potential around an isolated vortex line is shown in Figure 2, where the charge of the current carrier is chosen as positive, $q = |e|$, denoting that the charge carrier is hole. In Figure 1 we can see that the spatial gradient of $\delta\mu$ is about to be maximum when $r \sim \xi$. As a result, the column containing net negative charge in Figure 2 has a diameter of $r \sim \xi$. It is also easily seen in Figure 1 that when $\xi < r < \lambda$, $\delta\mu_{v3}$ is much greater than $\delta\mu_{v1}$ and $\delta\mu_{v2}$, and will not fade out before $r \sim \lambda$. As a result, the positive "shell" around the negative column in Figure 2 has a diameter of $r \sim \lambda$. This is caused by the supercurrent around the vortex and is just the so-called Bernoulli effect in London's discussions [1].

When $\kappa \gg 1$, the charge of the vortex per unit length has the order of magnitude like

$$Q \sim \frac{\hbar^2}{8mq}\frac{1}{\xi^2}. \tag{40}$$

For the vortex pancake in YBCO whose length is $d = 1.17 \times 10^{-9}$ m and $\xi = 1.91$ nm, the charge $Qd \sim 10^{-3}e$. This is similar to the results in the papers before [7].

6 The correction to GL equations

Finally we consider the correction that should be made to GL equations. Since the additional electrostatic potential (22) is yielded, the energy density of the electric field should be counted into the GL free energy density (16). The additional free energy density is

$$\mathscr{F}_{ad} = \frac{1}{8\pi}|\nabla\varphi|^2. \tag{41}$$

We should deduce the corrected GL equations by taking the variational derivative of total free energy

$$\mathscr{G} = \int (\mathscr{F} + \mathscr{F}_{ad})d^3r \tag{42}$$

where $\mathscr{F}$ is the normal GL free energy density shown in (16). We define $\psi(\mathbf{r})$ as

$$\psi(\mathbf{r}) = \frac{\Delta(\mathbf{r})}{|\Delta_\infty|} \tag{43}$$

and rewrite $\mathscr{F}$ in (16) as

$$\mathscr{F} = \frac{H_c^2}{4\pi}\left[-|\psi|^2 + \frac{1}{2}|\psi|^4 + \xi^2\left|\left(-i\nabla - \frac{2e}{\hbar c}\mathbf{A}\right)\psi\right|^2\right] + \frac{h^2}{8\pi} \tag{44}$$

where H_c is the critical field. In BCS theory, it is written as

$$H_c = -\frac{4\pi A^2}{B} = \frac{\hbar^2 c^2}{8e^2\xi^2\lambda^2}. \tag{45}$$

So the normal GL equations are written as

$$\frac{H_c^2}{4\pi}\left[-\psi + |\psi|^2\psi + \xi^2\left(-i\nabla - \frac{2e}{\hbar c}\mathbf{A}\right)^2\psi\right] = 0 \tag{46a}$$

$$\mathbf{j} = \frac{H_c^2\xi^2}{4\pi}\left[\frac{2e}{i\hbar}(\psi^*\nabla\psi - \psi\nabla\psi^*) - \frac{8e^2}{c\hbar^2}\psi^*\psi\mathbf{A}\right] \tag{46b}$$

rewrite the electrostatic potential φ as

$$\varphi \simeq P_1(1-\psi^2) + P_2\left|\left(-i\nabla - \frac{2e}{\hbar c}\mathbf{A}\right)\psi\right|^2 \tag{47}$$

where constant P_1 and P_2 are defined as

$$P_1 \equiv -\frac{1}{2q}\frac{\mathscr{C}}{E_F}\Delta_\infty^2\mathscr{R} \tag{48}$$

$$P_2 \equiv -\frac{\hbar^2}{4mq}\left(1 - \frac{T}{T_c}\right). \tag{49}$$

The variation respect to ψ^* is

$$\delta\mathscr{G} = \int \delta\mathscr{F}d^3r + \frac{1}{8\pi}\int \delta|\nabla\varphi|^2d^3r = \int \delta\mathscr{F}d^3r + \frac{1}{4\pi}\int \nabla^2\varphi\delta\varphi d^3r. \tag{50}$$

The variation to the additional electrostatic potential is therefore written as

$$\delta\varphi = \left[-P_1\psi + P_2\left(-i\nabla - \frac{2e}{\hbar c}\mathbf{A}\right)^2\psi\right]\delta\psi^*. \tag{51}$$

So (46a) becomes

$$\frac{H_c^2}{4\pi}\left[-\left(1 + \frac{P_1\nabla^2\varphi}{H_c^2}\right)\psi + |\psi|^2\psi + \xi^2\left(1 + \frac{P_2\nabla^2\varphi}{\xi^2H_c^2}\right)\left(-i\nabla - \frac{2e}{\hbar c}\mathbf{A}\right)^2\psi\right] = 0. \tag{52}$$

We know that $\nabla^2\varphi \sim \varphi/\xi^2$. With same argument (46b) becomes

$$\mathbf{j} = \frac{H_c^2\xi^2}{4\pi}\left(1 + \frac{P_2\nabla^2\varphi}{\xi^2H_c^2}\right) \times \left[\frac{2e}{i\hbar}(\psi^*\nabla\psi - \psi\nabla\psi^*) - \frac{8e^2}{c\hbar^2}\psi^*\psi\mathbf{A}\right]. \tag{53}$$

Apparently, we have both $\frac{P_1\nabla^2\varphi}{H_c^2} \ll 1$ and $\frac{P_2\nabla^2\varphi}{\xi^2H_c^2} \ll 1$. It is easily seen that the corrections made to the normal GL equations (46) are relatively small. The form of the corrected GL equations (52, 53) are very close to the original GL equations (46a, 46b). Only some very small corrections are added properly into corresponding items.

7 Conclusion

We have studied the problem of electrostatic potential distribution in superconductors where the order parameter is inhomogeneous. Unlike previous works, our theory is not based on a phenomenological theory, but is deduced from the *microscopic* theory of superconductors. We have introduced an approximation method to the BG self-consistent theory, which is similar to the TF approximation normally used in quantum mechanics in solving many-body problems. With this TF-like approximation, we brought out the distribution of electrostatic potential, which can be regarded as the first order approximation of the change of self-consistent field δU in BG equations. The electrostatic potential consists of three parts: the difference of absolute value of order parameter, the spatial gradient of the module and of the phase of the order parameter. All three parts have the similar order of magnitude and should not be neglected. Using the electrostatic potential we deduced, we estimated the value of the corresponding charge density of an isolated vortex in type-II superconductors, and the value is similar to the results in earlier papers. Considering the additional free energy density of the electrostatic field, corrections to the GL equations are made. It is shown that the corrections to GL equations are relatively small. And the structure of GL equations are not changed, with all corrections added properly to the corresponding items. So the electrostatic potential we derived is a good approximation to the self-consistent solution of BG equations.

Appendix A: The calculation of local chemical potential

The density of charge carrier in superconducting state is written as

$$\rho_s = \int_0^\infty [|u_k|^2 f_k + |v_k|^2(1-f_k)]\mathscr{D}(\varepsilon)d\varepsilon \tag{A.1}$$

where $f_k = 1/(e^{\beta E_k}+1)$. Substitute (8) into ρ_s, we have

$$\rho_s = \int_0^\infty \frac{1}{2}\left[1 - \frac{\varepsilon-\mu_s}{E_k}(1-2f_k)\right]\mathscr{D}(\varepsilon)d\varepsilon \tag{A.2}$$

where $E_k = \sqrt{(\varepsilon-\mu_s)^2 + |\Delta|^2}$. Now we define

$$x \equiv \varepsilon - \mu_s \tag{A.3a}$$

$$F(x) \equiv \frac{1}{2}\left[1 - \frac{x}{\sqrt{x^2+|\Delta|^2}}(1-2f_k)\right] \tag{A.3b}$$

$$G(x) \equiv \int_{-\infty}^{x} \mathscr{D}(x+\mu_s)dx \tag{A.3c}$$

and re-write ρ_s as

$$\begin{aligned}\rho_s &= \int_{-\mu_s}^{\infty} F(x)G'(x)dx \\ &= -\int_{-\mu_s}^{\infty} F'(x)G(x)dx + F(x)G(x)\Big|_{-\mu_s}^{\infty}.\end{aligned} \tag{A.4}$$

Normally we can take $\mu_s \gg \Delta$, so $F(x)G(x)|_{-\mu_s}^{\infty} \simeq 0$. According to reduced BCS theory, $\Delta(x)$ is nonzero only when $-\hbar\omega_D \le x \le \hbar\omega_D$. Thus

$$\rho_s \simeq -\int_{-\hbar\omega_D}^{\hbar\omega_D} F'(x)G(x)dx. \tag{A.5}$$

When $T \to 0$, since $f_k \to 0$, F(x) has the simple form

$$F(x) = \frac{1}{2}\left(1 - \frac{x}{\sqrt{x^2+|\Delta|^2}}\right) \tag{A.6}$$

and its differential quotient is

$$F'(x) = -\frac{1}{2}\frac{|\Delta|^2}{(x^2+|\Delta|^2)^{3/2}}. \tag{A.7}$$

Expand $G(x)$ at $x=0$, take the first three items

$$G(x) = G(0) + G'(0)x + \frac{1}{2}G''(0)x^2 + \cdots \tag{A.8}$$

and bring it into ρ_s, we have

$$\rho_s \simeq -\int_{-\hbar\omega_D}^{\hbar\omega_D}\left[G(0) + G'(0)x + \frac{1}{2}G''(0)x^2\right]F'(x)dx. \tag{A.9}$$

Normally we have $\hbar\omega_D \gg \Delta$, so the first item of (A.9) is approximately $G(0)$. Since $F'(x)$ is even, the second item is 0. In the third item we have

$$\begin{aligned}G''(0) &= \left.\frac{d\mathscr{D}(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=\mu_s} \\ &\simeq \left.\frac{d\mathscr{D}(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=\mu_n} = \frac{d\mathscr{D}_F}{d\varepsilon}\end{aligned} \tag{A.10}$$

where $\mathscr{D}_F = \mathscr{D}(\mu_n)$. Hence ρ_s is expressed as

$$\begin{aligned}\rho_s &\simeq G(0) + \frac{1}{2}\frac{d\mathscr{D}_F}{d\varepsilon}\int_{-\hbar\omega_D}^{\hbar\omega_D} F'(x)x^2dx \\ &\simeq G(0) - \frac{1}{2}\frac{d\mathscr{D}_F}{d\varepsilon}|\Delta|^2\left(\ln\frac{2\hbar\omega_D}{|\Delta|} - 1\right).\end{aligned} \tag{A.11}$$

On the other hand, the density of charge carrier in normal state at $T=0$ is written as

$$\begin{aligned}\rho_n &= \int_0^{\mu_n}\mathscr{D}(\varepsilon)d\varepsilon = G(\mu_n - \mu_s) \\ &= G(0) + \mathscr{D}_F(\mu_n - \mu_s).\end{aligned} \tag{A.12}$$

If we take $\rho_n \simeq \rho_s$ and compare the results above, we will get

$$\begin{aligned}\mu_s - \mu_n &\simeq -\frac{1}{2\mathscr{D}_F}\frac{d\mathscr{D}_F}{d\varepsilon}|\Delta|^2\left(\ln\frac{2\hbar\omega_D}{|\Delta|} - 1\right) \\ &= -\frac{1}{2}\frac{d\ln\mathscr{D}_F}{d\varepsilon}|\Delta|^2\left(\ln\frac{2\hbar\omega_D}{|\Delta|} - 1\right).\end{aligned} \tag{A.13}$$

When $T > 0$, the integrations can not be analytically deduced. But we may use numerical method to finish the integration, and get the similar results. The change of chemical potential may be approximately written as

$$\mu_s - \mu_n \simeq -\frac{1}{2}\frac{d\ln\mathscr{D}_F}{d\varepsilon}|\Delta|^2\mathscr{R}(\Delta, T). \tag{A.14}$$

When $T \to 0$, Parameter $\mathscr{R}$ goes to

$$\mathscr{R} \to \ln\frac{2\hbar\omega_D}{|\Delta|} - 1. \tag{A.15}$$

When $T > 0$, the value of $\mathscr{R}$ will change, but it will have the similar order of magnitude.

Appendix B: Charge screening taken account into the local chemical potential

For superconductors where the order parameter is homogeneous, $\Delta = \Delta_0$, the chemical potential is written as

$$\mu_s(\rho, T, \Delta_0) = \mu_n(\rho, T) + \phi_s(\rho, T, \Delta_0) \quad \text{(B.1)}$$

where μ_n is the chemical potential of charge carrier in normal state, and $\phi_s = \mu_s - \mu_n$ is the change of chemical potential when the superconductor changes from normal state to superconducting state. When the system is inhomogeneous, we have $\rho(\mathbf{r}) = \rho + \delta\rho(\mathbf{r})$ and $\Delta = \Delta(\mathbf{r})$. For the small homogeneous subsystem we are discussing, we omit parameter T and write the local chemical potential μ_L as

$$\mu_L(\rho + \delta\rho(\mathbf{r}), \Delta(\mathbf{r})) = \mu_n(\rho + \delta\rho(\mathbf{r})) + \phi_L(\rho + \delta\rho(\mathbf{r}), |\Delta(\mathbf{r})|) \quad \text{(B.2)}$$

where ϕ_L is the corresponding change of chemical potential of the local system. Since the chemical potential of the whole system must be identical, we introduce $V(\mathbf{r})$ being electrostatic potential between the local subsystem and the subsystems deep inside and write

$$\mu_L(\rho + \delta\rho(\mathbf{r})) + \phi_L(\rho + \delta\rho(\mathbf{r}), |\Delta(\mathbf{r})|) + qV(\mathbf{r}) = \mu_n(\rho) + \phi_s(\rho, \Delta_0) \quad \text{(B.3)}$$

where q is the charge of the carrier. On the other hand, according to the approximation and discussion in Section 3, one may write

$$\mu_L(\rho + \delta\rho(\mathbf{r})) - \mu_n(\rho) \simeq \left(\frac{\partial \mu_n}{\partial \rho}\right)\delta\rho(\mathbf{r}) \quad \text{(B.4)}$$

$$\phi_L(\rho + \delta\rho(\mathbf{r}), |\Delta(\mathbf{r})|) - \phi_s(\rho, \Delta_0) \simeq \delta\phi(\rho, |\Delta(\mathbf{r})|, \Delta_0) \quad \text{(B.5)}$$

where (B.5) is the formal expression of equation (15) in Section 3. Thus (B.3) may be written as

$$qV(\mathbf{r}) + \left(\frac{\partial \mu_n}{\partial \rho}\right)\delta\rho(\mathbf{r}) + \delta\phi(\rho, |\Delta(\mathbf{r})|, \Delta_0) = 0 \quad \text{(B.6)}$$

or

$$\delta\rho(\mathbf{r}) = -\frac{1}{(\partial \mu_n/\partial \rho)}(qV(\mathbf{r}) + \delta\phi(\rho, |\Delta(\mathbf{r})|, \Delta_0)). \quad \text{(B.7)}$$

and, in principle, the charge distribution in superconductors could be written as

$$Q(\mathbf{r}) = q\delta\rho(\mathbf{r}). \quad \text{(B.8)}$$

According to Poisson equation, $V(\mathbf{r})$ is determined by

$$\begin{aligned}\nabla^2 V(\mathbf{r}) &= -4\pi Q(\mathbf{r}) \\ &= \frac{q^2}{(\partial \mu_n/\partial \rho)}(V(\mathbf{r}) + \frac{1}{q}\delta\phi(\rho, |\Delta(\mathbf{r})|, \Delta_0)). \quad \text{(B.9)}\end{aligned}$$

Notice that

$$\frac{q^2}{(\partial \mu_n/\partial \rho)} = \frac{1}{l_{cs}^2} \quad \text{(B.10)}$$

and l_{cs} is just the charge screening length that generally discussed in solid state physics. Equation (B.9) is rewritten as

$$l_{cs}^2 \nabla^2 V(\mathbf{r}) = V(\mathbf{r}) + \frac{1}{q}\delta\phi(\rho, |\Delta(\mathbf{r})|, \Delta_0). \quad \text{(B.11)}$$

The spatial gradient of $V(\mathbf{r})$ has the order of ξ^{-1} or λ^{-1}. Thus $l_{cs}^2 \nabla^2 V(\mathbf{r}) \sim l_{cs}^2 V(\mathbf{r})/\xi^2$. It is easily seen that $l_{cs} \ll \xi$, i.e. $l_{cs}^2/\xi^2 \ll 1$. So we may expand $V(\mathbf{r})$ as

$$V(\mathbf{r}) = V^{(0)}(\mathbf{r}) + V^{(1)}(\mathbf{r}) + V^{(2)}(\mathbf{r}) + \cdots \quad \text{(B.12)}$$

and compare the corresponding items and get

$$V^{(0)}(\mathbf{r}) + \frac{1}{q}\delta\phi(\rho, |\Delta(\mathbf{r})|, \Delta_0) = 0 \quad \text{(B.13)}$$

$$l_{cs}^2 \nabla^2 V^{(0)}(\mathbf{r}) = V^{(1)}(\mathbf{r}) \quad \text{(B.14)}$$

$$l_{cs}^2 \nabla^2 V^{(1)}(\mathbf{r}) = V^{(2)}(\mathbf{r}) \quad \text{(B.15)}$$

$$\cdots\cdots$$

We may rewrite $Q(\mathbf{r})$ in equation (B.8) as

$$Q(\mathbf{r}) = -\frac{1}{4\pi}\frac{1}{l_{cs}^2}(V^{(0)}(\mathbf{r}) + \frac{1}{q}\delta\phi(\rho, |\Delta(\mathbf{r})|, \Delta_0) + V^{(1)}(\mathbf{r}) + \cdots).$$

Now we bring the above results into this equation and get the part of charge distribution that is caused by the module change of order parameter

$$\begin{aligned}Q(\mathbf{r}) &= -\frac{1}{4\pi l_{cs}^2}(V^{(1)}(\mathbf{r}) + \cdots) \\ &\simeq -\frac{1}{4\pi l_{cs}^2} l_{cs}^2 \frac{-1}{q}\nabla^2 \delta\phi(\rho, |\Delta(\mathbf{r})|, \Delta_0) \\ &= -\frac{1}{4\pi q}\nabla^2 \delta\phi(\rho, |\Delta(\mathbf{r})|, \Delta_0). \quad \text{(B.16)}\end{aligned}$$

This is the formal expression of the results in Section 3. It is easily seen from the above discussion that the charge screening has been fully taken account into the results. In fact, the charge screening is deduced from the mean field theory. Thus when we use a TF-like approximation shown in Section 3, the charge screening will be more precisely considered.

Appendix C: Order parameter and magnetic field near the vortex core

The order parameter and the magnetic field around an isolated vortex in type-II superconductors has been well studied in many books. But in our case, not only the magnitude of the order parameter and the magnetic field, but also their first and second order spatial gradient near the vortex core are of importance to the distribution of net charge carrier caused by the additional electrostatic field. Now we give a detailed discussion to this problem. GL equations are written as

$$\alpha\psi + \beta|\psi|^2\psi - \frac{\hbar^2}{2m}\left(-i\nabla - \frac{2e}{\hbar c}\mathbf{A}\right)^2\psi = 0 \quad \text{(C.1a)}$$

$$\mathbf{j} = \frac{2e\hbar}{m}(\psi^*\nabla\psi - \psi\nabla\psi^*) - \frac{4e^2}{mc}\psi^*\psi\mathbf{A} \quad \text{(C.1b)}$$

take $\psi = \Delta(\rho, \theta, \mathbf{z})$, substitute (33) into (C.1), use $\nabla\times\mathbf{H} = \frac{4\pi}{c}\mathbf{j}$ and $-\alpha/\beta = |\psi_\infty|^2$, GL equations become

$$\frac{d^2f}{dr^2} + \frac{1}{r}\frac{df}{dr} + \left(v - \frac{1}{r}\right)^2 f + f - f^3 = 0 \quad \text{(C.2a)}$$

$$\frac{d^2v}{dr^2} + \frac{1}{r}\frac{dv}{dr} + \frac{1}{\kappa^2}\left(v - \frac{1}{r}\right)f^2 - \frac{v}{r^2} = 0 \quad \text{(C.2b)}$$

where $r = \rho/\xi$, $a = \frac{\Phi}{2\pi\xi}v$, and $\lambda/\xi = \kappa$. $\Phi_0 = \frac{hc}{2|e|}$ is the flux quantum. The boundary condition is

$$f \to 0, \quad v \to 0, \quad (r \to 0) \quad \text{(C.3a)}$$

$$f \to 1, \quad v \to \frac{1}{r}, \quad (r \to \infty). \quad \text{(C.3b)}$$

Thus we may expand $f(r)$ and $v(r)$ near $r = 0$ as

$$f(r) = f_1 r^1 + f_2 r^2 + f_3 r^3 + \cdots \quad \text{(C.4a)}$$

$$v(r) = v_1 r^1 + v_2 r^2 + v_3 r^3 + \cdots \quad \text{(C.4b)}$$

Substitute (C.4) into (C.3) and compare the coefficients of the items having the same order of r, we have

$$f_1, v_1 = c \quad \text{(C.5a)}$$

$$f_2 = v_2 = 0 \quad \text{(C.5b)}$$

where c is random constant, and

$$f_3 = -\frac{1}{8}(1 - 2v_1)f_1 \quad \text{(C.5c)}$$

$$v_3 = \frac{1}{8\kappa^2}f_1^2. \quad \text{(C.5d)}$$

Now we choose the form of $f(r)$ and $v(r)$. It is recommended by Tinkham [18] that

$$f(r) \simeq \tanh(\nu r) \quad \text{(C.6)}$$

is a good approximation to the solution of GL equation, where ν is a constant having order of 1, and should principally be decided by the variation of free energy. When $\rho \gg \xi$, the solution of London equation is a good approximation to the magnetic field

$$H = \frac{\Phi_0}{2\pi\lambda^2}K_0(\rho/\lambda) \quad \text{(C.7)}$$

where K_0 is the zero order modified Bessel function of second kind. Take the gauge $a(r) = \frac{1}{r}\int_0^r h(r)rdr$, then $v(r)$ is

$$v(r) = \frac{1}{r} - K_1\left(\frac{r}{\kappa}\right) \quad \text{(C.8)}$$

where K_1 is the first order modified Bessel function of second kind. But the magnetic field (C.7) is divergent when $r \to 0$, so the expression of (C.8) should be modified near the core of the vortex.

Introduce $x = \sqrt{r^2 + x_0^2}$

$$x \to x_0, \qquad (r \to 0) \quad \text{(C.9a)}$$

$$x \to r, \qquad (r \to \infty) \quad \text{(C.9b)}$$

where x_0 is an undetermined constant that should have the order of 1. If we replace r with x, $H(x)$ has the boundary conditions that meet the physical need

$$H(x) \to H(x_0), \qquad (r \to 0) \quad \text{(C.10a)}$$

$$H(x) \to H(r), \qquad (r \to \infty). \quad \text{(C.10b)}$$

Use the gauge and do the integration, $v(r)$ becomes

$$v(r) = \frac{1}{r}\left[1 - \frac{xK_1(x/\kappa)}{x_0K_1(x_0/\kappa)}\right]. \quad \text{(C.11)}$$

Now we choose (C.6) and (C.11) as heuristic solutions and substitute them into (C.4). We find

$$f_2 = 0, \quad v_2 = 0 \quad \text{(C.12)}$$

which means (C.6) and (C.11) are suitable approximate solutions to GL equations. We also have

$$f_1 = \nu \quad \text{(C.13a)}$$

$$f_3 = -\nu^3/3 \quad \text{(C.13b)}$$

and

$$v_1 = \frac{1}{2x_0\kappa}\frac{K_0(x_0/\kappa)}{K_1(x_0/\kappa)} \quad \text{(C.13c)}$$

$$v_3 = -\frac{1}{8\kappa^2x_0^2}. \quad \text{(C.13d)}$$

Use (C.5) and assume $\kappa \gg 1$, we solute

$$\nu \simeq \sqrt{3/8} \simeq 0.61 \quad \text{(C.14)}$$

$$x_0 \simeq \sqrt{8/3} \simeq 1.63. \quad \text{(C.15)}$$

We rewrite the results in coordinate $(r, \theta, \mathbf{z})$ as

$$f(r) = \tanh(\nu r/\xi) \quad \text{(C.16a)}$$

$$a(r) = \frac{\hbar c}{2e\xi}\frac{1}{r}\left[1 - \frac{yK_1(y/\lambda)}{y_0K_1(y_0/\lambda)}\right] \quad \text{(C.16b)}$$

$$H = \frac{\Phi_0}{2\pi\lambda^2}K_0(y/\lambda) \quad \text{(C.16c)}$$

where $y = \sqrt{y_0^2 + r^2}$, and $y_0 = x_0\xi$.

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